Explicit Demazure character formula for negative dominant characters

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Abstract

In this paper, we prove that for any semisimple simply connected algebraic group G, for any regular dominant character λ of a maximal torus T of G and for any element τ in the Weyl group W, the character $e^{\rho}\cdot char(H^0(X(\tau), \mathcal{L}_{\lambda-\rho}))$ is equal to the sum $\sum_{w\leq\tau} char(H^{l(w)}(X(w), \mathcal{L}_{-\lambda}))^*)$ of the characters of dual of the top cohomology modules on the Schubert varieties $X(w)$, w running over all elements satisfying $w \leq \tau$. Using this result, we give a basis of the intersection of the Kernels of the Demazure operators D_α using the sums of the characters of $H^{l(w)}(X(w), \mathcal{L}_{-\lambda})$, where the sum is taken over all elements w in the Weyl group W of G.

Keywords: Schubert varieties, Demazure operators, negative dominant characters.

1 Introduction

The following notations will be maintained throughout this paper.

Let $\mathbb{C}$ denote the field of complex numbers. Let G a semisimple, simply connected algebraic group over $\mathbb{C}$. We fix a maximal torus T of G and let $X(T)$ denote the set of characters of T. Let $W = N(T)/T$ denote the Weyl group of G with respect to T. Let R denote the set of roots of G with respect to T.

Let R^+ denote the set of positive roots. Let B^+ be the Borel sub group of G containing T with respect to R^+. Let $S = \{\alpha_1, \ldots, \alpha_l\}$ denote the set of simple roots in R^+, where l

is the rank of G. Let B be the Borel subgroup of G containing T with respect to the set of negative roots $R^- = -R^+$.

For $\beta \in R^+$ we also use the notation $\beta > 0$. The simple reflection in the Weyl group corresponding to α_i is denoted by s_{α_i}.

Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\mathfrak{h}$ be the Lie algebra of T. Let $\mathfrak{b}$ be the Lie algebra of B.

We have $X(T) \bigotimes \mathbb{R} = Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form of $\mathfrak{h}$.

The positive definite W-invariant form on $Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of the Lie algebra $\mathfrak{g}$ of G is denoted by $(\ , \ )$. We use the notation $\langle \ , \ \rangle$ to denote $\langle \nu, \alpha \rangle = \frac{2(\nu,\alpha)}{(\alpha,\alpha)}$.

Let $\leq$ denote the partial order on $X(T)$ given by $\mu \leq \lambda$ if $\lambda - \mu$ is a non negative integral linear combination of simple roots. We also say that $\mu < \lambda$ if $\mu \leq \lambda$, and $\mu \neq \lambda$.

We denote by $X(T)^+$ the set of dominant characters of T with respect to B^+. Let ρ denote the half sum of all positive roots of G with respect to T and B^+.

We denote by $X(T)^+_{reg}$ by the set of all regular dominant characters of T.

For any simple root α, we denote the fundamental weight corresponding to α by ω_α.

For $w \in W$, let $l(w)$ denote the length of w. We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

Let $w_0 \in W$ denote the longest element of the Weyl group.

For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w.

Let $\leq$ denote the Bruhat order on W. We also say that $w < \tau$ if $w \leq \tau$, and $w \neq \tau$.

When λ is dominant, we have $H^i(X(w), \mathcal{L}_\lambda) = (0)$, for all $w \in W$ and for all $i \geq 1$. Further, Demazure character formula gives the character of the T-module $H^0(X(w), \mathcal{L}_\lambda)$ for dominant characters λ of T.

Demazure character formula gives only the Euler characteristic of the line bundle $\mathcal{L}_\lambda$, when λ is not dominant (see [7, II, 14.18]) or (see theorem in page 617, [1]).

However, when λ is not dominant, the character of the individual cohomology modules $H^i(X(w), \mathcal{L}_\lambda)$ is not known explicitly.

It is a difficult problem to understand the explicit characters of the individual cohomology modules $H^i(X(w), \mathcal{L}_\lambda)$ for an arbitrary $\lambda \in X(T)$.

When λ is a regular dominant character of T, we have $H^i(X(w), \mathcal{L}_{-\lambda}) = (0)$ for all $i \neq l(w)$ (cf [2], corollary 4.1). Further, if $\langle \lambda, \alpha \rangle \geq 2$, $H^{l(w)}(X(w), \mathcal{L}_{-\lambda})$ is a non zero highest weight B-module (see [8], theorem 5.8 (1)).

So, we need to study the character of the T- module $H^{l(w)}(X(w), \mathcal{L}_{-\lambda})$ for any w, and for any regular dominant character λ of T.

The aim of this paper is to give a formula for the character of $H^{l(\tau)}(X(\tau), \mathcal{L}_{-\lambda})$ for any regular dominant character λ of T and for any $\tau \in W$, inductively using the characters of $H^{l(w)}(X(w), \mathcal{L}_{-\lambda})$, where w is running over elements of W satisfying $w < \tau$.

More precisely, we prove that

Theorem:

Let $\tau \in W$. Let λ be a regular dominant character of T.

Then, the sum $\sum_{w \leq \tau} char(H^{l(w)}(X(w), \mathcal{L}_{-\lambda}))^*)$ is equal to $e^{\rho} \cdot char(H^0(X(\tau), \mathcal{L}_{\lambda-\rho}))$.

Please see theorem (3.2) for a precise statement.

When $X(w)$ is Gorenstein, we relate two characters ψ_w and χ'_w of T which are involved in using Serre duality on the Schubert Variety $X(w)$.

For a precise statement, see corollary (3.3).

The characters are discussed in [10] for arbitrary G, and in [11] when G is of type A_n.

We give a basis of the intersections of the Kernels of the Demazure operators D_{α} using the sums of the characters of $H^{l(w)}(X(w), \mathcal{L}_{-\lambda})$, where the sum is running over $w \in W$.

For a precise statement, see corollary (3.4).

The organisation of the Paper is as follows:

Section 2 consists of Preliminaries. In Section 3, we prove the Theorem stated above and derive some useful consequences.

2 Preliminaries

We denote by U (resp. U^+) the unipotent radical of B (resp B^+). We denote by P_{α} the minimal parabolic subgroup of G containing B and s_{α}. Let L_{α} denote the Levi subgroup of P_{α} containing T. We denote by B_{α} the intersection of L_{α} and B. Then L_{α} is the product of T and a homomorphic image G_{α} of $SL(2)$ via a homomorphism $\psi : SL(2) \longrightarrow L_{\alpha}$. (cf. [7,II.1.1.4]).

We refer to [6] for notation and preliminaries on semisimple Lie algebras and their root systems.

We choose an ordering of positive roots. Let U_{α} denote the T- stable one dimensional root subgroup of G corresponding to a positive root α. We write U^+ as product $\Pi_{\alpha \in R^+} U_{\alpha}$ in the orderiong choosen as above.

For a fixed $w \in W$, the set of all positive roots α which are made negative by w^{-1} by $R^+(w^{-1})$.

Given a $w \in W$ the closure in G/B of the B orbit of the coset wB is the Schubert variety corresponding to w, and is denoted by $X(w)$.

For each $w \in W$, we can write the cell $C(w)$ in the Schubert variety $X(w)$ as product $\Pi_{\alpha\in R^+(w^{-1})}U_\alpha$ in the same ordering choosen as above.

Let X_α denote the coordinate function on U^+ correspoding to the root subgroup U_α of U^+.

For any character χ of B, we denote by $\mathbb{C}_\chi$ the one dimensional representation of B corresponding to χ.

We make use of following points in computing cohomologies.

Since G is simply connected, the morphism $\psi : SL(2) \longrightarrow G_\alpha$ is an isomorphism, and hence $\psi : SL(2) \longrightarrow L_\alpha$ is injective. We denote this copy of $SL(2)$ in L_α by $SL(2,\alpha)$ We denote by B'_α the intersection of B_α and $SL(2,\alpha)$ in L_α.

We also note that the morphism $SL(2,\alpha)/B'_\alpha \hookrightarrow L_\alpha/B_\alpha$ induced by ψ is an isomorphism.

Since $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ is an isomorphism, to compute the cohomology $H^i(P_\alpha/B, V)$ for any B- module V, we treat V as a B_α- module and we compute $H^i(L_\alpha/B_\alpha, V)$

We recall some basic facts and results about Schubert varieties. A good reference for all this is the book by Jantzen (cf [7, II, Chapter 14]).

Let $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \dots s_{\alpha_{i_n}}$ be a reduced expression for $w \in W$. Define

$$Z(w) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \dots \times P_{\alpha_{i_n}}}{B \times \dots \times B},$$

where the action of $B \times \dots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \dots \times P_{\alpha_{i_n}}$ is given by $(p_1, \dots, p_n)(b_1, \dots, b_n) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \dots, b_{n-1}^{-1} \cdot p_n \cdot b_n)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$. We denote by ϕ_w the birational surjective morphism $\phi_w : Z(w) \longrightarrow X(w)$.

We note that for each reduced expression for w, $Z(w)$ is smooth, however, $Z(w)$ may not be independent of a reduced expression.

Let $f_n : Z(w) \longrightarrow Z(ws_{\alpha_n})$ denote the map induced by the projection $P_{\alpha_1} \times P_{\alpha_2} \times \dots \times P_{\alpha_n} \longrightarrow P_{\alpha_1} \times P_{\alpha_2} \times \dots \times P_{\alpha_{n-1}}$. Then we observe that f_n is a $P_{\alpha_n}/B \simeq \mathbf{P}^1$-fibration.

Let V be a B-module. Let $\mathcal{L}_w(V)$ denote the pull back to $X(w)$ of the homogeneous vector bundle on G/B associated to V. *By abuse of notation* we denote the pull back of $\mathcal{L}_w(V)$ to $Z(w)$ also by $\mathcal{L}_w(V)$, when there is no cause for confusion. Then, for $i \geq 0$, we have the following isomorphisms of B-linearized sheaves

$$R^i f_{n*} \mathcal{L}_w(V) = \mathcal{L}_{ws_{\alpha_n}}(H^i(P_{\alpha_n}/B, \mathcal{L}_w(V)).$$

This together with easy applications of Leray spectral sequences is the constantly used tool in what follows. We term this the *descending 1-step construction.*

We also have the *ascending 1-step construction* which too is used extensively in what follows sometimes in conjunction with the descending construction. We recall this for the convenience of the reader.

Let the notations be as above and write $\tau = s_\gamma w$, with $l(\tau) = l(w) + 1$, for some simple root γ. Then we have an induced morphism

$$g_1 : Z(\tau) \longrightarrow P_\gamma/B \simeq \mathbf{P}^1,$$

with fibres given by $Z(w)$. Again, by an application of the Leray spectral sequences together with the fact that the base is a $\mathbf{P}^1$, we obtain for every B-module V the following exact sequence of P_γ-modules:

$$(0) \longrightarrow H^1(P_\gamma/B, R^{i-1}g_{1*}\mathcal{L}_w(V)) \longrightarrow H^i(Z(\tau), \mathcal{L}_\tau(V)) \longrightarrow H^0(P_\gamma/B, R^i g_{1*}\mathcal{L}_w(V)) \longrightarrow (0).$$

This short exact sequence of B-modules will be used frequently in this paper. So, we denote this short exact sequence by *SES* when ever this is being used.

We also recall the following well-known isomorphisms:

- $\phi_{w*}\mathcal{O}_{Z(w)} = \mathcal{O}_{X(w)}$.
- $R^q\phi_{w*}\mathcal{O}_{Z(w)} = 0$ for $q > 0$.

This together with [7, II. 14.6] implies that we may use the Bott-Samelson schemes $Z(w)$ for the computation and study of all the cohomology modules $H^i(X(w), \mathcal{L}_w(V))$. Henceforth in this paper we shall use the Bott-Samelson schemes and their cohomology modules in all the computations.

Simplicity of Notation If V is a B-module and $\mathcal{L}_w(V)$ is the induced vector bundle on $Z(w)$ we denote the cohomology modules $H^i(Z(w), \mathcal{L}_w(V))$ by $H^i(w, V)$.

In particular if λ is a character of B we denote the cohomology modules $H^i(Z(w), \mathcal{L}_\lambda)$ by $H^i(w, \lambda)$.

2.0.1 Some constructions from Demazure's paper

We recall briefly two exact sequences from [4] that Demazure used in his short proof of the Borel-Weil-Bott theorem (cf. [3]). We use the same notation as in [4].

Let α be a simple root and let $\lambda \in X(T)$ be a weight such that $\langle \lambda, \alpha \rangle \geq 0$. For such a λ, we denote by $V_{\lambda,\alpha}$ the module $H^0(P_\alpha/B, \mathcal{L}_\lambda)$. Let $\mathbb{C}_\lambda$ denote the one dimensional B-module.

Here, we recall the following lemma due to Demazure on a short exact sequence of B -modules: (to obtain the second sequence we need to assume that $\langle \lambda, \alpha \rangle \geq 2$).

Lemma 2.1.

$$(0) \longrightarrow K \longrightarrow V_{\lambda,\alpha} \longrightarrow \mathbb{C}_\lambda \longrightarrow (0).$$
$$(0) \longrightarrow \mathbb{C}_{s_\alpha(\lambda)} \longrightarrow K \longrightarrow V_{\lambda-\alpha,\alpha} \longrightarrow (0).$$

A consequence of the above exact sequences is the following crucial lemma, a proof of which can be found in [4].

Lemma 2.2. *1. Let $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. If $\langle \lambda, \alpha \rangle \geq 0$ then $H^j(\tau, \lambda) = H^j(w, V_{\lambda,\alpha})$ for all $j \geq 0$.*

2. Let $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. If $\langle \lambda, \alpha \rangle \geq 0$, then $H^i(\tau, \lambda) = H^{i+1}(\tau, s_\alpha \cdot \lambda)$. Further, if $\langle \lambda, \alpha \rangle \leq -2$, then $H^i(\tau, \lambda) = H^{i-1}(\tau, s_\alpha \cdot \lambda)$.

3. If $\langle \lambda, \alpha \rangle = -1$, then $H^i(\tau, \lambda)$ vanishes for all $i \geq 0$ (cf. Prop 5.2(b), [7]).

3 Character of $(H^{l(w)}(w, -\lambda))^*$

In this section, we describe the character of the T-module $(H^{l(\tau)}(\tau, -\lambda))^*$ in terms of $e^\rho \cdot char(H^0(\tau, \lambda - \rho))$ and the sums of characters of $(H^{l(w)}(w, -\lambda))^*$, where the sum is taken over all elements w of W satisfying $w < \tau$.

So, in particular, we obtain the character of the T-module $H^{l(\tau)}(\tau, -\lambda)$.

Let D_w denote the boundary divisor of $X(w)$. For abuse of notation, we denote the sheaf on $X(w)$ corresponding to the Weil divisor D_w by D_w. Choose a non zero section $s \in H^0(X(w), D_w)$ such that the zero locus $Z(s)$ is D_w.

Let μ be a dominant character of T.

We now cosider the restriction map:

$res_w : H^0(w, \mu) \longrightarrow H^0(D_w, \mathcal{L}_\mu)$.

Let Ker_w denote the kernel of res_w.

Let ϵ_w (resp. ϵ_w^*) denote the character of the T- module Ker_w (resp. $(Ker_w)^*$).

Let $\tau \in W$. Let $h^0(\tau, \mu)$ (resp. $(h^0(\tau, \mu))^*$) denote the character of the T- module $H^0(\tau, \mu)$ (resp. $(H^0(\tau, \mu))^*$).

We now prove the following lemma giving a description of the character $h^0(\tau, \mu)$:

Lemma 3.1. *For any $\tau \in W$ and for any dominant character μ of T, we have $\sum_{w \leq \tau} \epsilon_w = h^0(\tau, \mu)$.*

Proof. We fix a $\tau \in W$.

Since the cell $C(\tau) = \Pi_{\alpha \in R^+(\tau^{-1})} U_\alpha$ is an open subset of $X(\tau)$, the restriction map $H^0(\tau, \mu) \longrightarrow H^0(C(\tau), \mathcal{L}_\mu)$ is injective.

Since $C(\tau)$ is the affine space $\Pi_{\alpha \in R^+(\tau^{-1})} U_\alpha$, the restriction of the line bundle $\mathcal{L}_\mu$ to $C(\tau)$ is trivial.

Thus, $H^0(\tau, \mu)$ is a subspace of $\mathbb{C}[X_\alpha : \alpha \in R^+(\tau^{-1})]$.

So, every section $s \in H^0(\tau, \mu)$ associates a polynomial f_s in the variables $\{X_\alpha : \alpha \in R^+(\tau^{-1})\}$.

Now, for every section $s \in H^0(\tau, \mu)$, there is a unique minimal element $w \leq \tau$ with respect to the Bruhat order on W such that the restriction f_s of s to $C(\tau)$ is a polynomial in the variables $\{X_\alpha : \alpha \in R^+(w^{-1})\}$.

Hence, for every $s \in H^0(\tau, \mu)$, there is a unique minmal $w \leq \tau$ such that $s \in Ker_w$.

Thus, the character of the B- module $H^0(\tau, \mu)$ is equal to the sum $\bigoplus_{w \leq \tau} \epsilon_w$ of the characters of the dual modules Ker_w, w running over all elements of W satisfying $w \leq \tau$.

Hence, we have $\sum_{w \leq \tau} \epsilon_w = h^0(\tau, \mu)$. This completes the proof of lemma.

□

Let λ be a regular dominant character of T. That is λ satisfies $\langle \lambda, \alpha \rangle \geq 1$ for each simple root α.

We recall notation from section 2: For $w \in W$, we denote by $H^{l(w)}(w, -\lambda)$ the top cohomology of the line bundle $\mathcal{L}_{-\lambda}$ on $X(w)$ associated to $-\lambda$.

Let e^ρ denote the element of the representation ring $\mathbb{Z}[X(T)]$ of T corresponding to ρ. Here, we use exponential notation e^ρ for using multiplication in the ring $\mathbb{Z}[X(T)]$.

Let $h^{l(w)}(w, -\lambda)$ denote the character of the T- module $H^{l(w)}(w, -\lambda)$.

Let $(h^{l(w)}(w, -\lambda))^*$ denote the character of the dual $H^{l(w)}(w, -\lambda)^*$ of the T- module $H^{l(w)}(w, -\lambda)$.

We have the following theorem:

Theorem 3.2. *For any $\tau \in W$, we have $\sum_{w \leq \tau} (h^{l(w)}(w, -\lambda))^* = e^\rho \cdot h^0(\tau, \lambda - \rho)$.*

Proof. Let $w \in W$ be such that $w \leq \tau$.

Let $\omega_{X(w)}$ denote the dualising sheaf on $X(w)$.

Observation 1 :

By Serre duality, the B modules $(H^{l(w)}(w, -\lambda))^*$ and $H^0(w, \mathcal{L}_\lambda \otimes \omega_{X(w)}) \bigotimes \mathbb{C}_{\psi_w}$ are isomorphic for some $\psi_w \in X(T)$.

This character ψ_w is discussed in [10].

On the other hand, we have $\omega_{X(w)} = -(\rho + D_w)$, where D_w is the boundary of $X(w)$. So, we have $H^0(X(w), \mathcal{L}_\lambda \otimes \omega_{X(w)}) = H^0(w, \lambda - \rho - D_w)$

Choose a non zero section $s \in H^0(X(w), D_w)$ such that the zero locus $Z(s)$ is D_w.

By excercise [5, II, 1.19], this section induces the following short exact sequence of sheaves on $X(w)$:

$$(0) \longrightarrow \mathcal{O}(-D_w) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{D_w} \longrightarrow (0).$$

Hence, we have the following exact sequence of B-modules:

$$(0) \longrightarrow H^0(w, \lambda - \rho - D_w) \bigotimes \mathbb{C}_{\chi_w} \longrightarrow H^0(w, \lambda - \rho) \longrightarrow H^0(D_w, \lambda - \rho) \longrightarrow (0).$$

Here, the $\mathbb{C}$ linear map $H^0(w, \lambda - \rho - D_w) \otimes \mathbb{C}_{\chi_w} \longrightarrow H^0(w, \lambda - \rho)$ is induced by the multiplication by s, and the $\mathbb{C}$- linear map $H^0(w, \lambda - \rho) \longrightarrow H^0(D_w, \lambda - \rho)$ is the restriction map, and we denote it by res_w. We note that res_w is a homorphism of B- modules.

Let Ker_w denote the kernel of res_w.

From the above short exact sequence, we see that the character of the B- module Ker_w is the same as that of $H^0(w, \lambda - \rho - D_w) \bigotimes \mathbb{C}_{\chi_w}$.

Hence, the character of $H^0(w, \lambda - \rho - D_w)$ is equal to $e^{-\chi_w} \cdot \epsilon_w$.

Using *Observation 1*, we see that the character of the T- module $(H^{l(w)}(w, -\lambda))^*$ is the same as that of $H^0(w, \lambda - \rho - D_w) \bigotimes \mathbb{C}_{\psi_w}$.

Thus, we have

Observation 2: The character $h^{l(w)}(w, -\lambda)^*$ of $(H^{l(w)}(w, -\lambda))^*$ is equal to $e^{\psi_w - \chi_w} \cdot \epsilon_w$.

We now show that $\psi_w - \chi_w = \rho$.

Since $H^{l(w)}(w, -\lambda)$ is a highest weight B-module with highest weight $w(-\lambda + \rho) - \rho$, its dual $(H^{l(w)}(w, -\lambda))^*$ is a lowest weight B-module with lowest weight $w(\lambda - \rho) + \rho$.

On the other hand, $H^0(w, \lambda - \rho)$ is a lowest weight module with lowest weight $w(\lambda) - w(\rho)$.

Hence, Ker_w is a lowest weight module with lowest weight $w(\lambda) - w(\rho)$.

Using *Observation 2*, we have

$w(\lambda - \rho) + \rho = w(\lambda) - w(\rho) + \psi_w - \chi_w$.

Hence, we have $\rho = \psi_w - \chi_w$.

Using *Observation 2*, we have

Observation 3 :

The character $h^{l(w)}(w, -\lambda)^*$ is equal to $e^{\rho} \cdot \epsilon_w$.

Now, taking $\mu = \lambda - \rho$ in lemma(3.1), we have $\sum_{w \leq \tau} \epsilon_w = h^0(\tau, \lambda - \rho)$.

Thus, using *Observation 3*, we see that

$\sum_{w \leq \tau} (h^{l(w)}(w, -\lambda))^* = \sum_{w \leq \tau} e^{\rho} \cdot \epsilon_w = e^{\rho} \cdot h^0(\tau, \lambda - \rho)$.

This completes the proof of the theorem.

$\square$

Let $w \in W$ be such that $X(w)$ is Gorenstein. Let χ'_w be the character of T such that $\mathcal{L}_{-2\rho+\chi'_w}$ is the canonical line bundle on $X(w)$.

When G is of type A_n, χ'_w is described in a nice combinatorial way in theorem 2, page 209 of [11].

Then, by Serre duality, there is a character ψ_w of T such that the B- modules $H^{l(w)}(w, -\lambda)^*$ and $H^0(w, \lambda - 2\rho + \chi'_w) \bigotimes \mathbb{C}_{\psi_w}$ are isomorphic.

With notation as above, the following corollary relates the two characters χ'_w and ψ_w of T.

Corollary 3.3. *Then, we have $\psi_w = \rho + w(\rho) - w(\chi'_w)$.*

Proof. By Serre duality, the B modules $(H^{l(w)}(w, -\lambda))^*$ and $H^0(w, \lambda - 2\rho + \chi'_w) \bigotimes \mathbb{C}_{\psi_w}$ are isomorphic.

Hence, the lowest weights of these two B- modules are the same.

Thus, we have $w(\lambda - \rho) + \rho = w(\lambda - \rho) - w(\rho) + w(\chi'_w) + \psi_w$.

Hence, we have $\psi_w = \rho + w(\rho) - w(\chi'_w)$.

□

Let D_α denote the Demazure operator on $\mathbb{Z}[X(T)]$ corresponding to a simple root α.

We recall from [7, II, 14.17] that

$$D_\alpha(e^\lambda) = \frac{e^\lambda - e^{s_\alpha(\lambda)-\alpha}}{1 - e^{-\alpha}}.$$

Let N_α denote the kernel of D_α.

Let N denote the intersection $\bigcap_{\alpha \in S} N_\alpha$ of the kernels of all D_α's, α running over all simple roots.

Then, we have

Corollary 3.4. *$\{\sum_{w \in W} h^{l(w)}(w, -\lambda) : \lambda \in X(T)^+_{reg}\}$ forms a $\mathbb{Z}$ basis for N.*

Proof. We first show that N consinsts of all $v \in \mathbb{Z}[X(T)]$ such that $D_\alpha(e^\rho v) = e^\rho v$ for each simple root $\alpha \in S$.

We fix a simple root α. Let $v \in \mathbb{Z}[X(T)]$ be such that $D_\alpha(v) = 0$.

Let $\mu \in X(T)$ be such that the coefficient of e^μ in the expression of v is non zero but the coefficient of $e^{\mu'}$ in the expression of v is zero for every $\mu' > \mu$ in the dominant ordering.

Since $D_\alpha(v) = 0$, using lemma(2.2), we see that either $\langle \mu, \alpha \rangle = -1$ and the coefficient of $e^{\mu-\alpha}$ in the expression of v is zero or $\langle \mu, \alpha \rangle \geq 0$ and the coefficient of $e^{\mu - i\alpha}$ in the expression of v is non zero for every $i = 0, 1, \cdots, \langle \mu, \alpha \rangle, 1 + \langle \mu, \alpha \rangle$.

Let $t = \langle \mu, \alpha \rangle$.

With out loss of generality, we may assume that the coefficient of e^{μ} in the expression of v is a positive integer, say a.

Proof of the case $t = -1$ is quite simple. So, we may assume that $t \geq 0$.

Using lemma(2.2), we see that $\sum_{i=0}^{t+1} e^{\mu - i\alpha}$ is in the Kernel of D_{α}. Hence, we have $D_{\alpha}(v - a(\sum_{i=0}^{t+1} e^{\mu - i\alpha})) = 0$. Hence, by induction on the dominant ordering on $X(T)$, we have

$$D_{\alpha}(e^{\rho}(v - \sum_{i=0}^{t+1} e^{\mu - i\alpha})) = e^{\rho}(v - \sum_{i=0}^{t+1} e^{\mu - i\alpha}).$$

On the other hand, $D_{\alpha}(e^{\rho}(\sum_{i=0}^{t+1} e^{\mu - i\alpha})) = e^{\rho}(\sum_{i=0}^{t+1} e^{\mu - i\alpha})$.

Thus, we have $D_{\alpha}(e^{\rho} v) = e^{\rho} v$.

Since $\alpha \in S$ was arbitrary, we must have $N = \{v \in \mathbb{Z}[X(T)] : D_{\alpha}(e^{\rho} v) = e^{\rho} v \; for \; all \; \alpha \in S\}$.

On the other hand, $D_{\alpha}(v') = v'$ for every simple root α if and only if v' is an integral linear combination of $h^0(w_0, \mu)$, μ running over dominant characters of T.

Thus, we have

Observation 1 :

The subset $\{e^{-\rho} h^0(w_0, \mu) : \mu \in X(T)^+\}$ forms a basis for N.

By theorem(3.2), we have

$\sum_{w \in W} (h^{l(w)}(w, -\lambda))^* = e^{\rho} h^0(w_0, \lambda - \rho)$.

The assertion of corollary follows from *Observation 1*.

$\square$

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